Italian Domination in Splitting Graph of a Graph

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Abstract

An Italian dominating function of a graph is a function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex $v \in V$ with $f(v) = 0, \sum_{u \in N(v)} f(u) \geq 2$. The weight of an Italian dominating function is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of an Italian dominating function f is called the Italian domination number, denoted by $\gamma_I(G)$. In this paper, we establish certain bounds for Italian domination number of splitting graphs of a graph and determine the exact value of the Italian domination number of the splitting graph of the path P_n.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E . An open neighborhood of a vertex v in V, denoted by $N(v)$, is the set of all vertices adjacent to v. Where as the set of all vertices adjacent to v and including v forms a closed neighborhood of a vertex v in V, denoted by $N[v]$. A dominating set of G is a subset $D \subseteq V$ where every vertex in V either belongs to D or is adjacent to at least one vertex in D.

An Italian dominating function (IDF) on G is a function $f : V \to \{0, 1, 2\}$, which induces a partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$, such that every vertex in V_0 is adjacent to at least two vertices in V_1 or to at least one vertex in V_2.

The weight of an Italian dominating function f is defined as the sum of weight of all vertices of G. The minimum weight among all Italian dominating functions on G is called the Italian domination number of G, denoted by $\gamma_I(G)$. An Italian dominating function that attains this minimum weight is called a γ_I-function.

If every γ_I-function on G has its range contained in $\{0, 1\}$ (equivalently, $V_2 = \emptyset$), then G is called an $I1$ graph [10].

Chellali et al. initiated the study of a variant of Roman dominating functions called the Roman $\{2\}$ dominating function, also known as the Italian dominating function in [4]. Henning et al. studied the Italian domination number of trees in[8].The impact of vertex addition on Italian domination number is discussed in [9]. Italian domination for various graph classes like Sierpinski Graphs and Mycielskian graphs have been studied in [12] and [13]

2 Preliminaries and Basic Definitions

The Splitting Graph $S_p(G)$ of a graph G is obtained by introducing a new vertex v' for each vertex v of G such that $N(v)$ in G is same as $N(v')$ in $S_p(G)$. The vertex v' is called the a twin vertex of v in $S_p(G)$. A vertex with degree 0 is called an isolated vertex. A vertex with degree 1 is called a leaf. A vertex adjacent to a leaf in G is called a stem. Sampathkumar et al. introduced Splitting graph in [11]. Deepalakshmi et al. studied various domination parameters on splitting graph in[5]. Roman domination number of splitting graphs was obtained by Atakul et al. in [1]. Varghese et al.[9] defined a true twin v' as a twin vertex with $N[v]$ in G is same as $N(v')$ in $S_p(G)$ and a false twin as a twin vertex with $N(v)$ in G is same as $N(v')$ in $S_p(G)$. In this paper a twin vertex denotes a false twin. For any other definitions and terminology the readers may refer to [2].The following results are useful in our paper.

Theorem 2.1. *[4] For every graph G, $\gamma(G) \leq \gamma_{\{R2\}}(G)$*

Theorem 2.2. *[5] Let G be a connected nontrivial graph. Then $\gamma(S_p(G)) = \gamma_t(G)$*

Theorem 2.3. *[7] For every graph G, $\gamma(G) \leq \gamma_t(G)$*

Theorem 2.4. *[9] Let u and u' be false twins in a graph G. Then there exists a γ_I-function f of G such that $f(u') \neq 2$.*

Observation 1 [6] Let G be a graph with $\gamma(G) = \gamma_I(G)$, and $f = (V_0, V_1, V_2)$ be an Italian dominating function. Then:

(a) $V_2 = \emptyset$, and $\gamma(G) = \gamma_2(G) = |V_1|$.

(b) $\delta(G) \geq 2$ and V_1 is an independent set.

(c) If $u \in V_0$ for some $u \in V$, then $|N(u) \cap N(v)| \geq 2$.

3 Main Results

Theorem 3.1. *Let G be a graph and $f = (V_0, V_1, V_2)$ a γ_I-function of G. Then $\gamma_I(G) \leq \gamma_I(S_p(G)) \leq \gamma_I(G) + |V_2| + |V_1|$.*

Proof. Let $f' = (V_0', V_1', V_2')$ be a γ_I-function of $S_p(G)$. Define $g : V(G) \to \{0, 1, 2\}$ as $g(v) = min\{f'(v) + f'(v'), 2\}$. Then g is an Italian dominating function of G. Therefore $\gamma_I(G) \leq w(g) \leq \gamma_I(S_p(G))$. To prove the right inequality, let $f = (V_0, V_1, V_2)$ be a γ_I-function of G. Define $g' : V(S_p(G)) \to \{0, 1, 2\}$ as

$$g'(u) = \begin{cases} f(u), & \text{if } u \in V_2 \cup V_1 \\ 1, & \text{if } u = v' \text{ and } v \in V_2 \cup V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly g' is an Italian dominating function of $S_p(G)$ with $w(g') = \gamma_I(G) + |V_2| + |V_1|$. Hence, the theorem. □

Theorem 3.2. *For any graph G without isolated vertices, $\gamma(G) + 1 \leq \gamma_I(S_p(G)) \leq 3\gamma(G)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_I-function of $S_p(G)$ and D be a minimum dominating set in $S_p(G)$. To prove the left inequality, if possible assume $\gamma(G) = \gamma_I(S_p(G))$. Then we have, $\gamma(G) = \gamma(S_p(G)) = \gamma_I(S_p(G))$ by Theorems 2.1, 2.2, 2.3 and 3.1. Then by Observation 1, $V_2 = \phi$. Thus V_1 becomes a dominating set of $S_p(G)$ such that every vertex of $S_p(G)$ is adjacent to at least two vertices of V_1. Then there exists a v in V_1 such that $D' = V_1 - \{v\}$ is also a dominating set of $S_p(G)$. This contradicts the minimality of D. To prove the right inequality, define a function $f' : V(G) \to \{0, 1, 2\}$ as

$$f'(u) = \begin{cases} 2, & \text{if } u \in D, \\ 1, & \text{if } u = v' \text{ and } v \in D, \\ 0, & \text{otherwise.} \end{cases}$$

f' is an Italian dominating function of $S_p(G)$ with $w(f') = 3\gamma(G)$. □

Theorem 3.3. *Let G be a graph with $n \geq 3$ vertices and $\delta(G) \geq 2$. Then $\gamma_I(Sp(G)) \leq n$.*

Proof. Define a function f as follows

$$f(v) = \begin{cases} 1, & \text{if } v \in V(G) \\ 0, & \text{otherwise.} \end{cases}$$

Clearly f is an Italian dominating function with $w(f) = n$ since any $v \in V(S_p(G)) - V(G)$ is adjacent to at least two vertices of $V(G)$. □

Theorem 3.4. *Let G be a graph without isolated vertices having n vertices, l leaves and s stem vertices. Then $\gamma_I(Sp(G)) \leq n - l + 2s$.*

Proof. We define a function f in $V(S_p(G))$ by assigning the value 1 to every vertex in G except for the leaves and the stem vertex of G; assigning the value 2 to every stem vertex of G and 1 to every twin vertex of the stem vertex . All other vertices carry the weight 0 in $S_p(G)$. Clearly f is an Italian dominating function of G with $w(f) = n - l + 2s$. □

Proposition 3.5. *Let $G = (V, E)$ be a graph. Then there exist a γ_I function $f = (V_0, V_1, V_2)$ for $S_p(G)$ such that V_2 contains no pendant vertex v of G.*

Proof. Let v be a pendant vertex of G where uv is the pendant edge. Then v can Italian dominate at most u, u', v in $S_p(G)$. If f is a γ_I function of $S_p(G)$ with $f(v) = 2$, then we can reassign $f(u) = f(u') = 1$ and $f(v) = 0$. Which is again a γ_I function of $S_p(G)$. □

Proposition 3.6. *Let $n \in N$ and $n \neq 2, 5$ then $n = 3k_1 + 4k_2$ where $k_1, k_2 \geq 0$.*

Proof. Any $n \in N$ can be written as $3k$ or $3k+1$ or $3k+2$. If $n = 3k$ the proof is trivial. If $n = 3k+1$ then $n = 3(k-1)+4 = 3k_1+4$ and if $n = 3k+2$ then $n = 3(k-2)+8 = 3k_1+4k_2$. The proof holds $\forall\ n \in N$ and $n \neq 2, 5$. □

Theorem 3.7. *For a path P_n where $n \neq 2, 5$, $\gamma_I(S_p(P_n)) = n$.*

Proof. We can verify the cases for $3 \leq n \leq 9, n \neq 5$.
We use induction on n to prove for $n \geq 10$. Suppose that $\gamma_I(S_p(P_k)) = k$ for $10 \leq k \leq n-1$. By Proposition 3.6 if $n \in N$ and $n \neq 2, 5$ we have $n = 3k_1 + 4k_2$. Therefore $\gamma_I(S_p(P_n)) \leq \gamma_I(S_p(P_{3k_1})) + \gamma_I(S_p(P_{4k_2})) = 3k_1 + 3k_2 = n$.
To prove the reverse inequality, let $f = (V_0, V_1, V_2)$ be a γ_I-function of $S_p(P_n)$. By Theorem 2.4 and Theorem 3.5, $v_n, v'_n, v'_{n-1} \in V_0 \cup V_1$.

Case 1: $f(v_n) = f(v'_n) = 0$.

Case 1 A : Let $f(v'_{n-1}) = 0$, then $f(v_{n-1}) = f(v_{n-2}) = 2$. Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} min(2, f(v_{n-3} + f(v'_{n-3}) + f(v_{n-5}) & \text{if } u = v_{n-5} \\ 0, & \text{if } u = v_{n-3}, v'_{n-3} \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$ with $w(g) = w(f)$. Clearly $g|_{V(S_P(P_{n-4})}$ is an Italian dominating function for $S_P(P_{n-4})$ with $\gamma_I(S_p(P_{n-4}) \leq w(g|_{V(S_P(P_{n-4}))})$. Hence $n-4 \leq w(g|_{V(S_P(P_{n-4}))})$.
$n \leq w(g|_{V(S_P(P_{n-4}))}) + 4 = w(g) = w(f) = \gamma_I(Sp(P_n))$.

Case 1 B : Let $f(v'_{n-1}) = 1$. Then $f(v_{n-1}) = 2$. Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} min(2, f(v_{n-4} + f(v'_{n-2}) + f(v_{n-2}) & \text{if } u = v_{n-4} \\ 0, & \text{if } u = v_{n-2}, v'_{n-2} \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$ with $w(g) = w(f)$. Clearly $g|_{V(S_P(P_{n-3})}$ is an Italian dominating function for $S_P(P_{n-3})$ with $\gamma_I(S_p(P_{n-3}) \leq w(g|_{V(S_P(P_{n-3}))})$. Hence $n-3 \leq w(g|_{V(S_P(P_{n-3}))})$.
$n \leq w(g|_{V(S_P(P_{n-3}))}) + 3 = w(g) = w(f) = \gamma_I(Sp(P_n))$.

Case 2: $f(v_n) = 0, f(v'_n) = 1$.

Case 2 A : Let $f(v'_{n-1}) = 0$, then $f(v_{n-1}) = 2$. Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} 0, & \text{if } u = v'_n \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$ with $w(g) < w(f)$. This case is not possible.

Case 2 B : Let $f(v'_{n-1}) = 1$. Then $f(v_{n-1}) \geq 1$. Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} 2 & \text{if } u = v_{n-1} \\ 0, & \text{if } u = v'_n \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$ with $w(g) = w(f)$. The proof is same as *Case 1B*

Case 3: $f(v_n) = 1, f(v'_n) = 1$.

Case 3 A : Let $f(v'_{n-1}) = 0$, then by minimality of f, $f(v_{n-2}) = 1$ and $f(v_{n-1}) = 0$. By proposition 2.4, $f(v'_{n-2}) \in \{0, 1\}$. Suppose $f(v'_{n-2}) = 0$, then $f(v_{n-3}) = 2$. Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} 0, & \text{if } u \in \{v_n, v'_n, v_{n-2}, v_{n-3}\}, \\ 1, & \text{if } u = v'_{n-1}, \\ 2, & \text{if } u = v_{n-1}, \\ \min(f(v) + 1, 2), & \text{if } u \in \{v_{n-4}, v'_{n-4}\}, \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$ with $w(g) = w(f)$. The proof is same as *Case 1B*. Suppose $f(v'_{n-2}) = 1$, then $f(v_{n-3}) = 2$ Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} 0, & \text{if } u \in \{v_n, v'_n, v'_{n-2}\}, \\ 2, & \text{if } u \in \{v_{n-1}, v_{n-2}\} \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$ with $w(g) = w(f)$. The proof is same as *Case 1A*.

Case 3 B : Let $f(v'_{n-1}) = 1$. Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} 2 & \text{if } u = v_{n-1} \\ 0, & \text{if } u \in \{v_n, v'_n\} \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$ with $w(g) = w(f)$. The proof is same as *Case 1B*.

Case 4: $f(v_n) = 1, f(v'_n) = 0$.

Case 4 A : Let $f(v'_{n-1}) = 0$, then $f(v_{n-1}) = 2$ and $f(v_{n-2}) \geq 1$. Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} 0, & \text{if } u = v_n, \\ 1, & \text{if } u = v'_{n-1}, \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$. The proof is same as *Case 1B*

Case 4 B : Let $f(v'_{n-1}) = 1$. Then $f(v_{n-1}) = 2$. Define $g : V(S_P(P_n)) \to \{0, 1, 2\}$ as,

$$g(u) = \begin{cases} 0, & \text{if } u = v_n \\ f(v), & \text{otherwise.} \end{cases}$$

g is a γ_I-function of $S_p(P_n)$ with $w(g) < w(f)$. This case is not possible.

□

Remark: For $n = 2, 5$, $\gamma_I(S_p(P_n)) = n + 1$.

4 Conclusion

In this paper, we have derived the bounds for the Italian domination number of splitting graphs in terms of various parameters like domination number and Italian domination number of the original graph. The exact Italian domination number of $S_p(P_n)$ is also derived. A study on the Italian domination number of the splitting graph of various graph classes is an interesting direction for future research.

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